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LONG RANGE TRAJECTORY PREDICTION ERRORS FOR LEAST SQUARES SMOOTHING

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By

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LONG RANGE TRAJECTORY PREDICTION ERRORS FOR LEAST SQUARES SMOOTHING

ABSTRACT

This paper presents exact formulas and graphs to determine the root mean-square error of prediction for position and velocity estimates of a target, based on least-squares smoothing. The prediction interval is taken from an end-point estimate up to ten times the smoothing interval. The r.m.s. values may be used to determine how large a search volume must be to accommodate a substantial data dropout interval.

1.0 INTRODUCTION

The application considered is as follows: Suppose one observes tracking data over M sample points, spaced T seconds apart, and smooths the data by curve fitting a least-squares polynomial of degree $n < M$ to the data. One wishes a "best" estimate of the position and velocity of the target at a time $TM(1 + \rho)$ as measured from the beginning of the data set. Values of ρ considered in this paper are shown in Figures 1 through 5 (See Appendix) for $\rho = 0$, .5 and 1 to 10. Thus if tracking data is obtained over say $M = 10$ samples for $T = 1.$ second sampling interval, and a data dropout occurs for ten seconds, one could curve fit a least-squares quadratic or cubic to the ten data points, and estimate the target's position and velocity ten seconds later. The r.m.s. of the estimate can be used to determine the size of the search volume to find the target.

The r.m.s. values must be considered as lower bounds to determine search volumes since bias errors, due to departure of the actual trajectory from the low-degree polynomical, will increase the prediction error.

The curves presented in this paper for the r.m.s. values are determined from exact equations. A set of asymptotic values of r.m.s. for $M >> 1$ and $\rho >> 1$ are also derived.

1.1 SOLUTION FOR MEAN SQUARE ERROR (δ^2)

Formulas for computing the r.m.s. for the output of a finite memory optimal-digital filter, when the input signal consists of a nonrandom polynomial of degree n plus uncorrelated stationary noise, have been published [1]. The optimal filter is equivalent to least-squares curve fitting of an n^{th} degree polynomial.

It is shown that:

$$\frac{\sigma_\Delta^2 \cdot T^{2K}}{\sigma_N^2} = \sum_{L=K}^{n} \frac{\left[\xi_L^{(K)}(M+\alpha)\right]^2}{S(L,M)} \equiv \delta^2 (M, \alpha, K, n) \qquad (1)$$

$$S(L,M) = \frac{(L!)^4 M \prod_{j=1}^{L} (M^2 - j^2)}{(2L)!\,(2L+1)!} \qquad (2)$$

where

T = interval between samples.

σ_N^2 = mean square of input noise $N(u)$.

M = number of samples in digital filter of fixed memory length $(M - 1)T$.

σ_Δ^2 = mean square of output of filter.

K = derivative of desired output.

$T\cdot(M+\alpha)$ = time at which desired output is evaluated.

$\xi_L^{(K)}(M+\alpha)$ = K^{th} derivative of the orthogonal polynomial of degree L evaluated at $t = (M+\alpha)\cdot T$

In the referenced paper, extensive curves δ are presented for $\alpha = 0$, or end-point prediction $\alpha = 1$, or next-point prediction and $\alpha = -[M-1]/2$, or smoothing of the mid-point of the data interval for 0^{th}, 1^{st}, and 2^{nd} derivative estimates. The computations for δ are extended for long-range prediction errors in the following.

Let the estimate be desired at the normalized (T=1) time

$$U = (1 + \rho) \cdot M$$

corresponding to $\alpha = M\rho$. Curves of δ vs M are presented for values of $0 \leq \rho \leq 10$, n = 1 to 3 for zero derivative estimates and, n = 2, 3, for first derivative estimates. It is easily shown that

$$\delta^2(M, M\rho, 0, 0) = \frac{1}{M} = \frac{1}{S(0, M)} \tag{3}$$

independent of ρ, and

$$\delta^2(M, M\rho, 1, 0) = 0$$

$$\delta^2(M, M\rho, 1, 1) = \frac{12}{M(M^2 - 1)} = \frac{1}{S(L, M)} \tag{4}$$

independent of ρ.

Asymptotic formulas are presented for δ as $M \to \infty$ and an approximation to δ are given for moderate values of M.

1.2 APPROXIMATE AND ASYMPTOTIC FORMULAS FOR δ^2

In this section, simplified equations for δ^2 will be derived for K = 0, and asymptotic formulas for K = 0 and K = 1.

The term S(L, M) can be written

$$S(L,M) = C_L \cdot M^{2L+1} \prod_{j=1}^{L} \left(1-\left(\frac{j}{M}\right)^2\right) \tag{5}$$

$$= C_L \, M^{2L+1} \left[1 - 0\left(\frac{L^3/2}{3M}\right)\right]$$

$$\cong C_L \, M^{2L+1} \tag{6}$$

for $\quad \frac{L^3/2}{2M} << 1$

where $\quad C_L = \frac{(L!)^4}{(2L)!\ (2L+1)!} \qquad (7)$

and $\quad C_0 = 1,\ C_1 = 1/12,\ C_2 = 1/180,\ C_3 = 1/2800. \qquad (8)$

Let $\quad \eta = (1/2+ \rho) \qquad (9)$

Then by substitution into the orthogonal polynomials, one may show that

$$\xi_0\ (U) = 1$$

$$\xi_1\ (U) = \left[U - \frac{M+1}{2}\right]$$

$$\xi_2\ (U) = \left[U - \frac{M+1}{2}\right]^2 - \left[\frac{M^2-1}{12}\right]$$

$$\xi_3\ (U) = \left[U - \frac{M+1}{2}\right]^3 - \left[U- \frac{M+1}{2}\right]\left\{\frac{3M^2-7}{20}\right\} \tag{10}$$

for $M >> 1 \quad \eta \geq 1$

$$\delta^2 (M, M\rho, 0, 0) = 1/M$$

$$\delta^2 (M, M\rho, 0, 1) \cong 1/M \left[\frac{\eta^2}{C_1} + 1\right]$$

$$\delta^2 (M, M\rho, 0, 2) \cong 1/M \left[\frac{\eta^4}{C_2} - 18\eta^2\right]$$

$$\delta^2 (M, M\rho, 0, 3) \cong 1/M \left[\frac{\eta^6}{C_3} - 660\eta^4\right] \qquad (11)$$

As η, $M \to \infty$ one has the asymptotic equations

$$\delta (M, M\rho, 0, L) \sim \frac{\eta^L}{\sqrt{MC_L}} \qquad L = 0, 1, 2, \ldots \qquad (12)$$

Similarly for first derivative estimation, the orthogonal polynomials evaluated at U are given by

$$\xi_1' = 1$$

$$\xi_2' = 2[U - \frac{M + 1}{2}]$$

$$\xi_3' = 3[U - \frac{M + 1}{2}]^2 - [\frac{3M^2 - 7}{20}]. \qquad (13)$$

For $M >> 1$, $\eta \geq 1$, an approximate formula for δ^2 is given by

$$\delta^2(M, M\rho, 1, 0) = 0$$

$$\delta^2(M, M\rho, 1, 1) \simeq \frac{12}{M^3}$$

$$\delta^2(M, M\rho, 1, 2) \simeq \frac{12}{M^3}[1 + 60\eta^2]$$

$$\delta^2(M, M\rho, 1, 3) \simeq \frac{1800\eta^2}{M^3}[14\eta^2 - 1]. \qquad (14)$$

For $M >> 1$, $\eta >> 1$ an asymptotic formula for δ is given by

$$\delta(M, M\rho, 1, L) \sim \frac{L\eta^{L-1}}{M\sqrt{MC_L}} \qquad L = 0, 1, 2, \ldots \qquad (15)$$

BIBLIOGRAPHY

[1] Blum, M. "On the Mean Square Noise Power of an Optimum Linear Discrete Filter Operating on Polynomial Plus White Noise Input," IRE Transactions of the Professional Group on Information Theory, Vol. IT-3, No. 4 (December 1957), pp. 225-231.

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APPENDIX

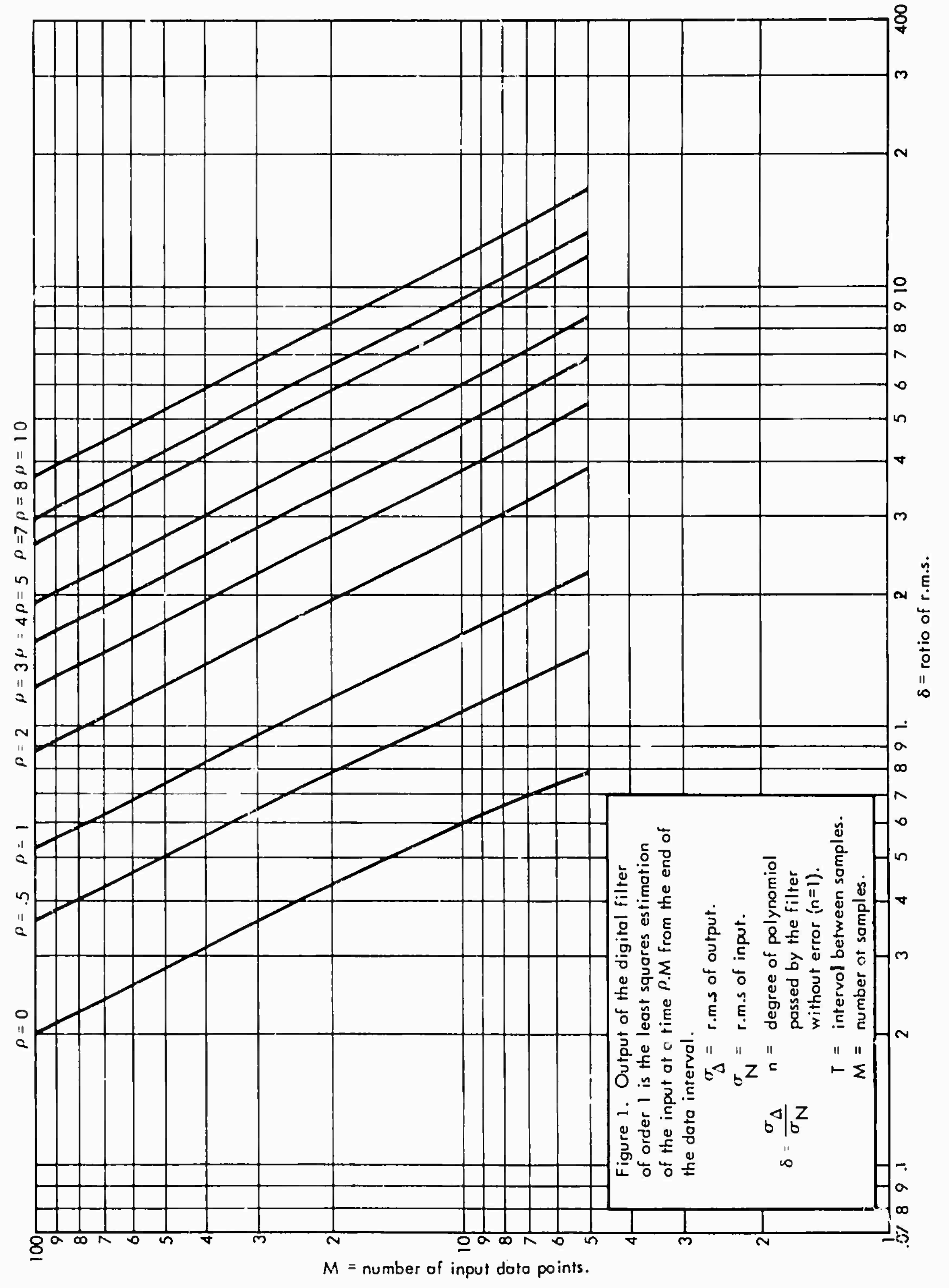

Figure 1. Output of the digital filter of order 1 is the least squares estimation of the input at a time ρ.M from the end of the data interval.
σ_Δ = r.m.s of output.
σ_N = r.m.s of input.
n = degree of polynomial passed by the filter without error (n=1).
T = interval between samples.
M = number of samples.
$\delta = \frac{\sigma_\Delta}{\sigma_N}$
ρ = 0
ρ = .5
ρ = 1
ρ = 2
ρ = 3
ρ = 4
ρ = 5
ρ = 7
ρ = 8
ρ = 10
M = number of input data points.
δ = ratio of r.m.s.

Figure 1

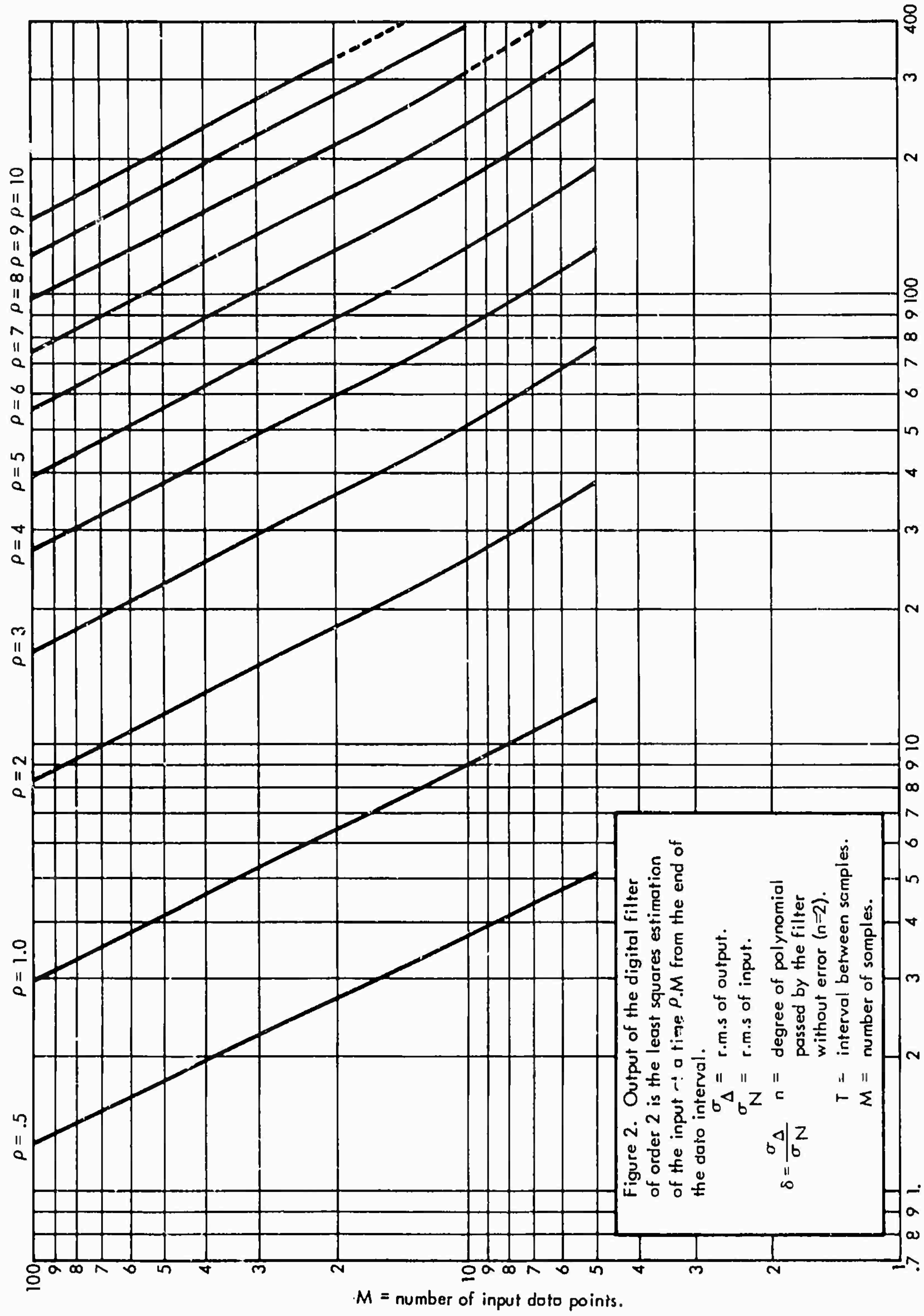

Figure 2. Output of the digital filter of order 2 is the least squares estimation of the input at a time ρ.M from the end of the data interval.

Figure 2

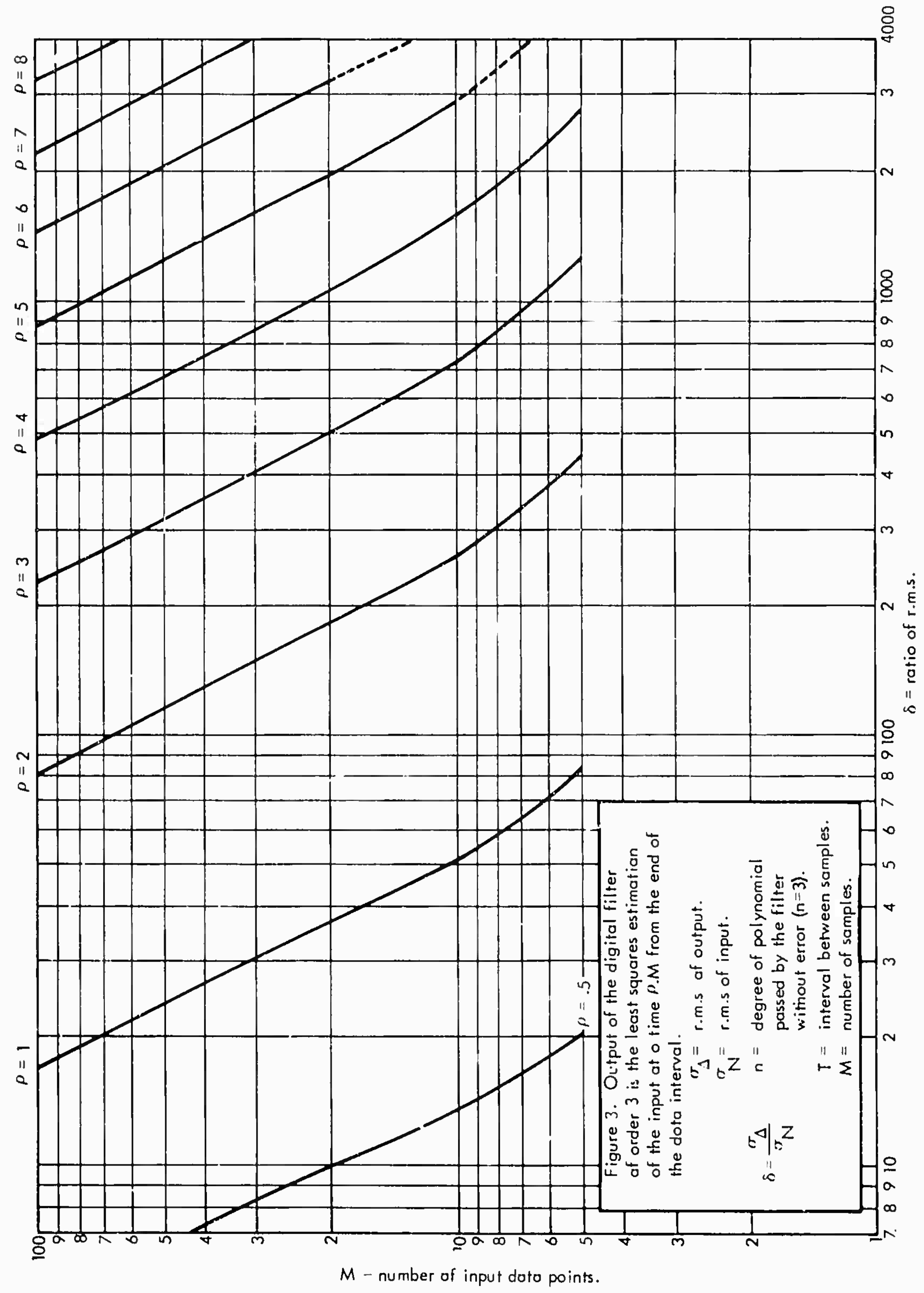
ρ = 1
ρ = 2
ρ = 3
ρ = 4
ρ = 5
ρ = 6
ρ = 7
ρ = 8
ρ = .5
Figure 3. Output of the digital filter of order 3 is the least squares estimation of the input at a time ρ.M from the end of the data interval.
σ_Δ = r.m.s of output.
σ_N = r.m.s of input.
$\delta = \frac{\sigma_\Delta}{\sigma_N}$
n = degree of polynomial passed by the filter without error (n=3).
T = interval between samples.
M = number of samples.
M – number of input data points.
δ = ratio of r.m.s.

Figure 3

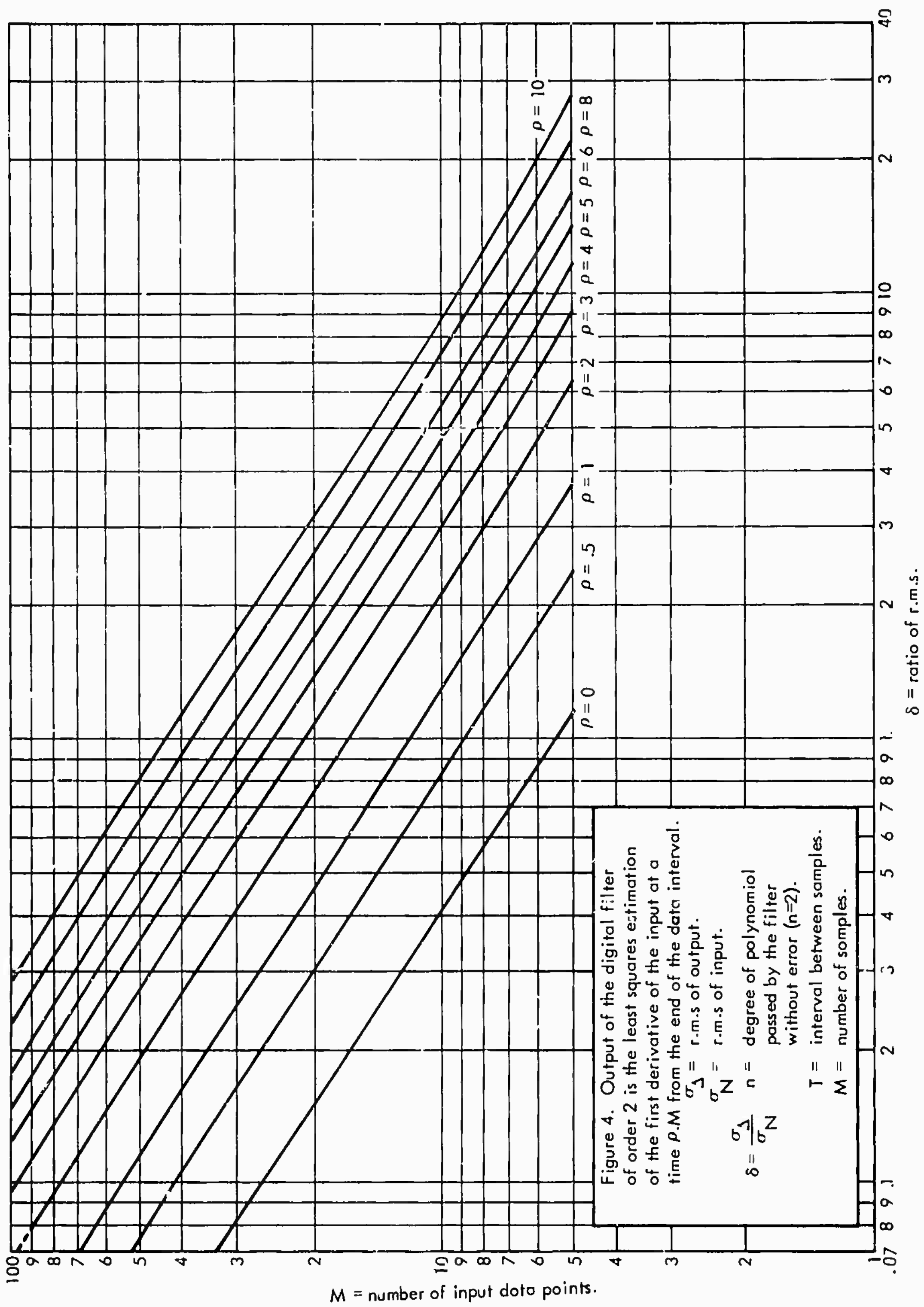
Figure 4. Output of the digital filter of order 2 is the least squares estimation of the first derivative of the input at a time ρ.M from the end of the data interval.
σ_Δ = r.m.s of output.
σ_N = r.m.s of input.
$\delta = \frac{\sigma_\Delta}{\sigma_N}$
n = degree of polynomial passed by the filter without error (n=2).
T = interval between samples.
M = number of samples.
ρ = 0
ρ = .5
ρ = 1
ρ = 2
ρ = 3
ρ = 4
ρ = 5
ρ = 6
ρ = 8
ρ = 10
M = number of input data points.
δ = ratio of r.m.s.

Figure 4

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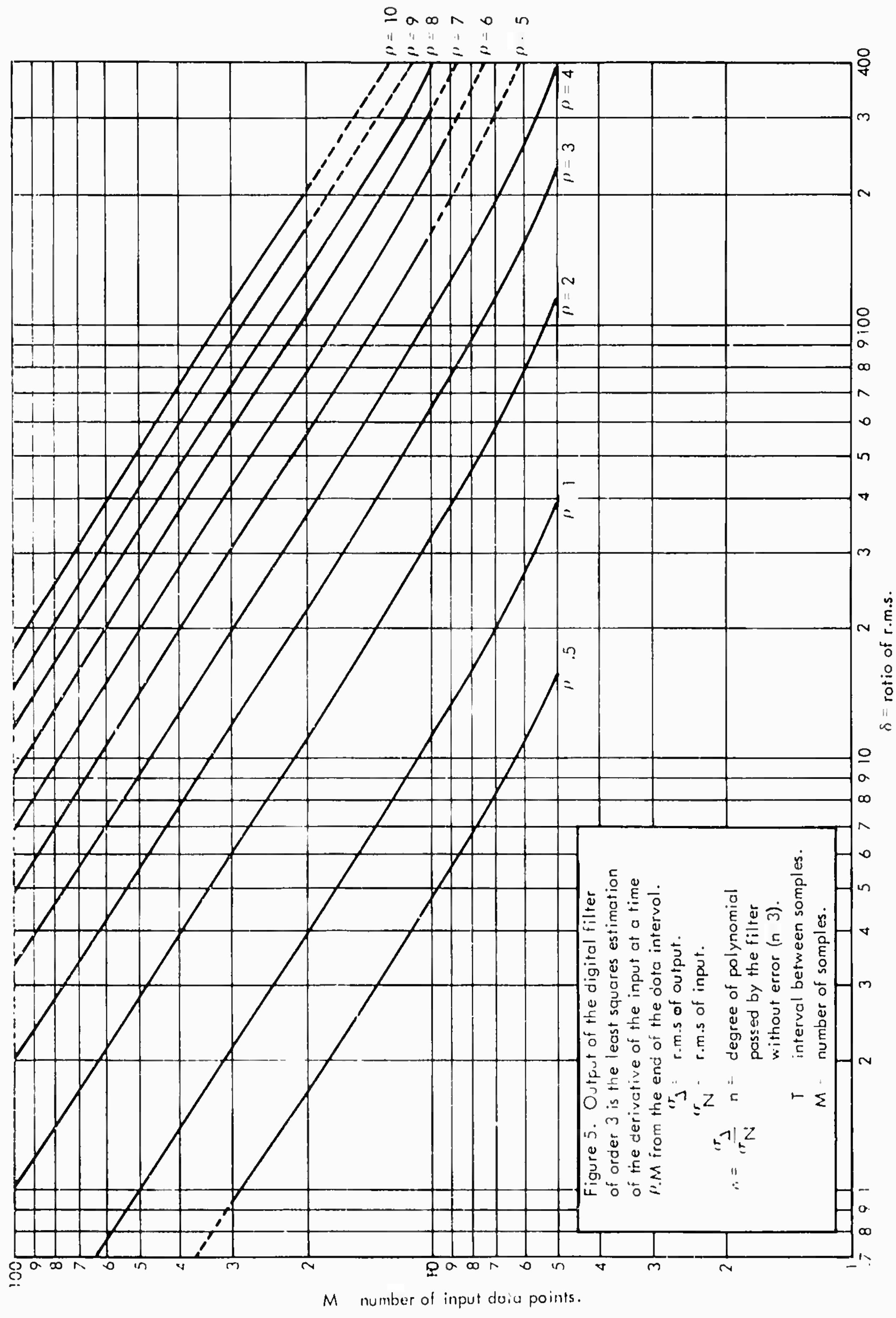

Figure 5

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